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The motion of a point mass along a string☆

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ABSTRACT

An equation for the trajectory of a point mass (a particle) when it moves (slides) in a plane by inertia along a weightless elastic thread (string), stretched between two fixed points is obtained. The time dependence of the trajectory parameter is established. An equation of the trajectory of the particle when it suddenly decelerates is obtained. Forced motion of the particle along a straight string (as a model of the swinging of a lift on an elastic tether in zero gravity) is considered.

This paper supplements Reference 1.

1. The equation of the trajectory of a particle sliding along a string

Consider the plane motion of a point mass (a particle) along an elastic weightless thread (a string), stretched between two fixed points.

In the plane of motion of the particle, we specify a fixed system of coordinates Oxy such that the points where the string are fastened lie on the x axis at a distance of unit length on different sides of the origin of coordinates.

Suppose r_1 and r_2 are the distances of the particle of unit mass from the right and left points of attachment respectively, g is the coefficient of tensile stiffness of the string (it is assumed that the strain of the string obeys Hooke's law) and Δ is the preliminary tension of the string.

After introducing the new coordinates q_1 and q_2 and the time τ using the formula

$$q_{1,2} = (r_1 \pm r_2)/2, \quad dt = \sqrt{q_1^2 - q_2^2}\,d\tau$$

and separating the variables in the Hamilton–Jacobi equation, the following relations were obtained in Ref. 1

$$q_2 = \sin[\sqrt{c_1}(\tau + \tau_0 - C_1)],$$
$$\frac{dq_1}{d\tau} = \sqrt{(q_1^2 - 1)(l_3 - c_1 + l_2 q_1 - l_1 q_1^2)} \tag{1.1}$$

where

$$l_1 = 4g, \quad l_2 = 4g(2-\Delta), \quad l_3 = 2\left(h - \frac{2g}{1-\Delta}\right)$$

h is the energy constant, and τ, c_1 and C_1 are arbitrary constants ($c_1 > 0$).

To represent the solution in real time it is necessary to invert the integral of the equation for q_1 (the second equality of (1.1)). (This was done approximately in Ref. 1 for the case when the particle moves with a small deviation from the x axis.)

We will write the second equality of (1.1) as follows:

$$\left(\frac{dq_1}{d\tau}\right)^2 = L(q_1),$$
$$L(q_1) = A_0 q_1^4 + 4A_1 q_1^3 + 6A_2 q_1^2 + 4A_3 q_1 + A_4 \tag{1.2}$$

where

$$A_0 = -l_1, \quad A_1 = \frac{1}{4}l_2, \quad A_2 = \frac{1}{6}(l_3 + l_1 - c_1),$$
$$A_3 = -A_1, \quad A_4 = -l_3 + c_1 \tag{1.3}$$

The general solution of Eq. (1.2) can be expressed in terms of the Weierstrass function $\mathcal{P}$, the invariants of which[2] are

$$g_2 = A_0 A_4 + 3A_2^2 - 4A_1 A_3,$$
$$g_3 = A_0 A_2 A_4 + 2A_1 A_2 A_3 - A_2^3 - A_0 A_3^2 - A_1^2 A_4 \tag{1.4}$$

and has the form

$$q_1 = a_1 + \frac{a_2}{\wp(\tau + C_1) - a_3} \tag{1.5}$$

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Here a_1 is some root of the polynomial $L(q_1)$; for example $a_1 = 1$, while a_2 and a_3 are formed from a_1 using the formulae[2]

$$a_2 = A_0 a_1^3 + 3A_1 a_1^2 + 3A_2 a_1 + A_3,$$
$$a_3 = \frac{1}{2}A_0 a_1^2 + A_1 a_1 + \frac{1}{2}A_2 \tag{1.6}$$

When $a_1 = 1$ we obtain

$$a_2 = \frac{1}{2}(-l_1 + l_2 + l_3 - c_1), \quad a_3 = \frac{1}{12}(-5l_1 + 3l_2 + l_3 - c_1)$$

The first equation of (1.1) and equality (1.5) enable us to determine the trajectory of the particle in parametric form, if the new time τ is considered as a parameter.

In order to express the solution in real time, we return to the relation

$$t = \int_0^\tau \sqrt{q_1^2 - q_2^2}\, d\tau \tag{1.7}$$

(the initial instant of time t_0 can be assumed to be equal to zero and $\tau_0 = 0$).

The solution in the finite interval $\tau \in [0, \tau^*]$, where τ^* does not exceed the half-period of the sine on the right-hand side of the left equality of (1.1), i.e. $\tau^*\ \pi/\sqrt{c_1}$, is of practical interest (when $\tau = \tau^*$ the direction of motion along the x axis reverses).

We will give an upper estimate of the time t of the motion of the particle for the current value of parameter $\tau \in [0, \tau^*]$.

Writing the radical on the right-hand side of Eq. (1.7) in the form $\sqrt{q_1 - q_2}\sqrt{q_1 + q_2}$, we apply the Holder inequaity[3] to integral (1.7) and obtain the limit

$$t \le \{[\tau + a_2 I_2(\tau)]^2 - I_1^2(\tau)\}^{1/2} \tag{1.8}$$

where

$$I_1(\tau) = \frac{1}{\sqrt{c_1}}\{\cos[\sqrt{c_1}(\tau + \tau_0 - C_1)] - \cos[\sqrt{c_1}(\tau_0 - C_1)]\}$$

$$I_2(\tau) = \int_0^\tau \frac{d\tau}{\wp(\tau + C_1) - a_3} = \int_{C_1}^{C_1+\tau} \frac{d\tau}{\wp(z) - \wp(z_3)} =$$
$$= -\frac{1}{\wp'(z_3)}[\zeta(z + z_3) - \zeta(z - z_3)]\Big|_{C_1}^{C_1+\tau} \tag{1.9}$$

where $z = C_1 + \tau$, z_3 is the solution of the equation $\mathcal{P}(z_3) = a_3$ and $\zeta(z)$ is the zeta function. (A method of solving the last equation uniquely can be found in Ref. 2.)

2. The equation of the trajectory of a particle fastened to a string

When the particle is suddenly decelerated (jammed), the potential energy of the string is represented by the formula[1]

$$V = g\left[2 + 2\frac{(x-a)^2 + y^2}{(1-a^2)(2-\Delta)} - \sqrt{(1-x)^2 + y^2} - \sqrt{(1+x)^2 + y^2}\right],$$
$$a \in [-1 + \varepsilon,\ 1 - \varepsilon]$$
$$0 < \varepsilon < 1$$

where a is the abscissa of the particle in the rest position. By Hooke's law $\Delta \ll 1$. We will therefore henceforth assume that $2 - \Delta \approx 2$.

In q_1, q_2 coordinates, we obtain the following expressions for the potential and kinetic energies

$$V = 2g\left[1 - q_1 + \frac{1}{2}\frac{a^2 - 1 + q_1^2 + q_2^2 - 2aq_1q_2}{1 - a^2}\right] \tag{2.1}$$

$$T = \frac{1}{2}\frac{(q_1^2 - 1)p_1^2 + (1 - q_2^2)p_2^2}{q_1^2 - q_2^2}; \quad p_1 = \frac{\partial T}{\partial \dot{q}_1}, \quad p_2 = \frac{\partial T}{\partial \dot{q}_2} \tag{2.2}$$

When $a = 0$, after changing to the new time τ, the variables in the Hamilton–Jacobi equation can be separated

$$\left(\frac{dq_1}{d\tau}\right)^2 = L_1(q_1), \quad \left(\frac{dq_2}{d\tau}\right)^2 = g_1 l_1 q_2^4 - (c_1 + g_1 l_1) q_2^2 - c_1 \tag{2.3}$$

The polynomial $L_1(q_1)$ has the same form as $L(q_1)$ in Eq. (1.2), but with different coefficients

$$A_0 = -g_1 l_1,\ A_1 = \frac{1}{4}g_1,\ A_2 = \frac{1}{6}(h_2 - c_1 + g_1 l_1),$$
$$A_3 = -A_1, \quad A_4 = c_1 - h_2$$
$$g_1 = 4g,\ l_1 = \frac{1}{2(1 - a^2)},\ h_2 = 2gh + (l_1 - 1)g_1\ (c_1 > 0)$$

The invariants g_2 and g_3 of the Weierstrass function have the form (1.4), while the solution has the form (1.5) with $a_1 = 1$ and coefficients a_2 and a_3 calculated from formulae (1.6).

The solution of the second equation of (2.3) also has the form (1.5).

Hence, we have obtained the trajectory of a jammed particle (in the non-local formulation of the problem). (The solution in real time can be expressed in the same way as in Section 1.)

If the initial velocity is directed along the y axis, the trajectory degenerates into a section of the y axis. In this case, from the energy integral

$$H = \frac{1}{2}\dot{y}^2 + g(3 - 2\sqrt{1 + y^2})$$

at the level $H = h$, after making the replacement $y^2 = z^2 - 1$, we obtain

$$t = \frac{1}{2\sqrt{2g}}\int_1^z \frac{z\,dz}{\sqrt{(z^2 - 1)[h/g - (z-1)^2]}} \tag{2.4}$$

The roots z_1 and z_2 of the trinomial in the radicand are real and have different signs, where $z_2 < 0$ and $1 < z_1$. The integral can be written in the form of a combination of elliptic integrals of the first and third kinds.[4]

Hence we obtain the oscillation period of the particle

$$T_0 = \frac{2}{g\sqrt{z_1 - z_2}}\left\{(z_1 + 1)\Pi\left(\frac{\pi}{2}, \frac{1 - z_1}{2}, k\right) - F\left(\frac{\pi}{2}, k\right)\right\};$$
$$k = \sqrt{\frac{(z_1 - 1)(z_2 + 1)}{2(z_1 - z_2)}}$$

If the motion of the particle occurs at a low energy level, the limitation $a = 0$ can be removed. In the linear approximation, the variables x and y can be separated and describe harmonic oscillators.

3. Forced motion of a particle along a string

We will consider the case of the forced motion of a particle with constant velocity about a point on a tensioned string, which is, at the given instant, in contact with the particle. The velocity is directed along the straight line passing through the points where the string is fastened $x=-1$, $x=1$.

The particle can mean a body – a lift, dynamically symmetrical about this straight line, the rollers of which rotate with constant angular velocity u, and at the instant of time $t=0$ at the point $x=a$, $a\in(-1;\ 1)$ squeeze the string (cable) without slipping. Suppose that, before squeezing, the lift slides along the string in the direction of x increasing with a velocity $v_0 \le u$.

From the instant it is squeezed the string is deformed, it is stretched through the rollers and acts on the lift with a force of elasticity $2g(ut-x+a)/(1-x^2)$, and hence the equation of motion of the particle (of unit mass), taking the dissipation force $(-2b\dot{x})$, into account, confining ourselves to considering the motion in the region of the point $x=a$, can be written in the form

$$\ddot{x} + 2b\dot{x} + \frac{2g}{1-a^2}x = \frac{2g}{1-a^2}(ut+a) \tag{3.1}$$

From its general solution (when $b^2<2g/(1-a^2)$), with the initial conditions $x(0)=a$, $\dot{x}(0)=v_0$, we obtain

$$x = e^{-bt}(C_1\cos\omega t + C_2\sin\omega t) + ut + a - (1-a^2)bu/g$$

$$C_1 = \frac{(1-a^2)bu}{q}, \quad C_2 = \frac{bC_1 + v_0 - u}{\omega}, \quad \omega = \sqrt{\frac{2g}{1-a^2} - b^2} \tag{3.2}$$

It can be seen from the solution that the particle begins to move (when $t>0$) with a velocity less than u_0 (this can be seen particularly well when $b=0$, when $C_1=0$ and $C_2<0$). For further motion at constant velocity oscillations are superimposed.

Solution (3.2) enables us to estimate the preliminary tension Δ, at which tension in the branches of the string does not disappear during motion,

$$\Delta \ge ut_M + a - x(t_M)$$

Here t_M is the time of the first maximum on the right-hand side of the inequality.

If the string is not pretensioned, then from the instant it is squeezed by the rotating rollers the left branch of the string (the part of the string between the left support and the lift) sags, and the force of elasticity will be defined by the expression $g[ut-(x-a)]/(1-x)$. Then, up to the instant when the tension in the right-hand branch disappears, one can use the solution of Eq. (3.1), with the quantity $2g$ replaced by g and the quantity $(1-a^2)$ replaced by $(1-a)$. The instant of time t_n when the tension in the right-hand branch disappears is given by the expression $ut-(x-a)=0$, after substituting the solution into it. The return of the string to the unperturbed state corresponds to this instant, and the velocity of the lift is a maximum. Further motion to the right support may continue in a sliding mode, while jamming of the string occurs in the last (deceleration) stage (see Section 2).

Note again that if a gravitational force acts along the string, a constant term, corresponding to the static strain of the string, is added to solution (3.2).

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